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The effect of combined vertical and horizontal heterogeneity on the onset of convection in a bidisperse porous medium

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Abstract

The effects of both horizontal and vertical hydrodynamic and thermal heterogeneity, on the onset of convection in a horizontal layer of a saturated bidisperse porous medium uniformly heated from below, are studied analytically using linear stability theory for the case of weak heterogeneity. It is found that the effect of such heterogeneity on the critical value of the Rayleigh number *Ra* based on mean properties is of second order if the properties vary in a piecewise constant or linear fashion. The effects of horizontal heterogeneity and vertical heterogeneity are then comparable once the aspect ratio is taken into account, and to a first approximation are independent. The thermal heterogeneity of the *p*-phase can be quite significant when the thermal diffusivity of that phase is large relative to that of the *f*-phase.

Keywords: Bidisperse porous media; Natural convection; Heterogeneity; Instability; Horton–Rogers–Lapwood problem

1. Introduction

The problem of the onset of convection in a horizontal layer of fluid heated uniformly from below is commonly called the Rayleigh-Bénard problem in the case of a fluid clear of solid material and the Horton–Rogers–Lapwood (HRL) problem for the case of a fluid-saturated porous medium. A feature of such convection is that it generally appears in the form of cells whose horizontal dimension is of the same order as their vertical dimension. The critical dimensionless wavenumber a_c in the linear stability analysis turns out to have a value of about 3 in most cases. In the HRL problem with conducting impervious boundaries $a_c = \pi$, a value that corresponds to rolls of square cross-section. An exception occurs in the case of "insulating" (with respect to perturbation heat flux) boundaries. For this case $a_c = 0$, so that the convection occurs as a single cell.

In recent discussions about the effect of heterogeneity (of either permeability or thermal conductivity or both) on convection in a porous medium it has been noted that in the case of strong heterogeneity there can be dramatic effects [1–3]. Even in the case of weak heterogeneity it is of interest to investigate the combined effects of vertical heterogeneity (property variation in the vertical direction, including horizontal layering as a special case) and horizontal heterogeneity. This is the subject of the analysis of Nield and Kuznetsov [4]. The survey of the effects of heterogeneity in Nield and Bejan [5] indicates this topic had not been considered previously. In their analytical study Nield and Kuznetsov [4] found that the effect of such heterogeneity on the critical value of the Rayleigh number *Ra* based on mean properties is of second order if the properties vary in a piecewise constant or linear fashion. The effects of horizontal heterogeneity and vertical heterogeneity are then comparable and to a first approximation are independent. For the case of conducting impermeable top and bottom

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Nomenclature

A	aspect ratio (height to width)
c	specific heat at constant pressure
d	height of the enclosure
Da_f	Darcy number, $\frac{\tilde{\mu} K_{f0}}{\mu d^2}$
g	gravity
h	inter-phase heat transfer coefficient (incorporating the specific area)
H	inter-phase heat transfer parameter, $\frac{hd^2}{\phi k_{f0}}$
k^*	overall (effective) thermal conductivity
k_f^*	thermal conductivity of the f-phase
k_p^*	thermal conductivity of the p-phase
k_{f0}	mean value of $k_f^*(x^*, y^*)$
k_{p0}	mean value of $k_p^*(x^*, y^*)$
$\hat{k}_f$	k_f^*/k_{f0}
$\hat{k}_p$	k_p^*/k_{p0}
K_r	permeability ratio, $\frac{K_{p0}}{K_{f0}}$
K^*	permeability
K_{f0}	mean value of $K_f^*(x^*, y^*)$
K_{p0}	mean value of $K_p^*(x^*, y^*)$
$\widehat{K}_f$	K_f^*/K_{f0}
$\widehat{K}_p$	K_p^*/K_{p0}
L	enclosure width
p	dimensionless pressure, $\frac{(\rho c)_f K_{f0}}{k_f \mu} p$
p^*	pressure
Ra_f	Rayleigh number, $\frac{\rho_F g \hat{\beta}(T_1 - T_u) K_{f0} d}{\mu \phi k_{f0}/(\rho c)_f}$
S	parameter defined by Eq. (48)
t^*	time
t	dimensionless time, $\frac{k_{f0}}{(\rho c)_f d^2} t^*$
T_f^*	temperature in the f-phase
T_p^*	temperature in the p-phase
T_1	temperature at the lower boundary
T_u	temperature at the upper boundary
T_F^*	volume average of the temperature over the fluid, defined by Eq. (3)
T_0	reference temperature, $T_1 - T_u$
u_f	dimensionless horizontal velocity in the f-phase, $\frac{(\rho c)_f d}{\phi k_{f0}} u_f^*$
u_p	dimensionless horizontal velocity in the p-phase, $\frac{(\rho c)_f d}{(1-\phi) k_{p0}} u_p^*$
u^*	dimensional horizontal velocity
$\mathbf{v}^*$	vector of Darcy velocity, (u^*, v^*)
v_f	dimensionless vertical velocity in the f-phase, $\frac{(\rho c)_f d}{\phi k_{f0}} v_f^*$
v_p	dimensionless vertical velocity in the p-phase, $\frac{(\rho c)_f d}{(1-\phi) k_{p0}} v_p^*$
v^*	dimensional vertical velocity
x	dimensionless horizontal coordinate, x^*/d
x^*	horizontal coordinate
y	dimensionless upward vertical coordinate, y^*/d
y^*	upward vertical coordinate

Greek symbols

α	thermal diffusivity ratio, $\frac{k_{f0}}{k_{p0}} \frac{(\rho c)_p}{(\rho c)_f}$
β	modified thermal diffusivity ratio, $\frac{(1-\phi) k_{p0} (\rho c)_f}{\phi k_{f0} (\rho c)_p}$
$\hat{\beta}$	volumetric thermal expansion coefficient of the fluid
γ	modified thermal conductivity ratio, $\frac{\phi k_{f0}}{(1-\phi) k_{p0}}$
ε	porosity in the p-phase
θ_f	dimensionless temperature in the f-phase, $\frac{T_f^* - T_u}{T_1 - T_u}$
θ_p	dimensionless temperature in the p-phase, $\frac{T_p^* - T_u}{T_1 - T_u}$
Θ_f	perturbation quality defined by Eq. (24)
Θ_p	perturbation quality defined by Eq. (24)
θ_F	parameter defined by Eq. (18)
$\tilde{\mu}$	effective viscosity of the porous medium
ρ_F	density of the fluid
σ_f	inter-phase momentum transfer parameter, $\frac{\zeta K_{f0}}{\mu}$
ϕ	volume fraction of the f-phase
ψ_f	streamfunction in the f-phase, defined by Eq. (14)
ψ_p	streamfunction in the p-phase, defined by Eq. (14)
Ψ_f	perturbation quality defined by Eq. (24)
Ψ_p	perturbation quality defined by Eq. (24)

Subscripts

f	f-phase
p	p-phase

Superscript

*	dimensional variable

boundaries and a square box, the effects of permeability heterogeneity and conductivity permeability each cause a reduction in the critical value of Ra, while for the case of a tall box there can be either a reduction or an increase. It was found by Nield and Kuznetsov [6] that in the case of a shallow box with constant-flux top and bottom boundaries there can be either a reduction of increase in the critical value of the Rayleigh number.

In the present paper the analysis of Nield and Kuznetsov [4] is extended in various ways. First, the momentum equation is extended from the Darcy to the Brinkman model. Second, the effect of local thermal non-equilibrium is included. Third, at the same time an extension is made from a regular porous medium to a bidisperse porous medium, or BDPM (a porous medium in which the "solid" phase is itself a porous medium), illustrated in Fig. 1a, using a two-velocity as well as a two-temperature model.

At each stage of the extension the complexity of the analysis is increased. In order to obtain significant results it is necessary to use a second order Galerkin expansion,

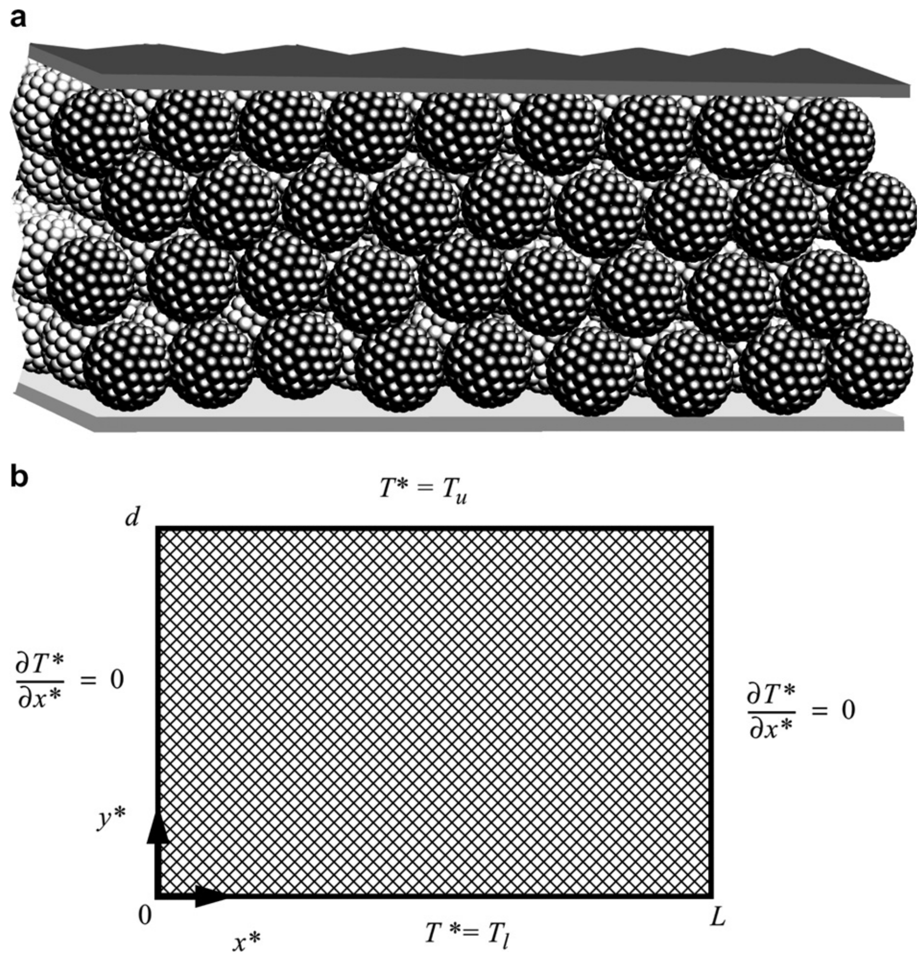

Fig. 1. (a) Conceptual diagram of a bidisperse porous medium and (b) schematic diagram of the problem (the walls are also assumed to be stress-free).

rather than just a first order one. In the case of trial functions involving two coordinates that means four trial functions for each primary dependent variable, and there are four such variables for the two-velocity and two-temperature model. Hence the analysis involves the algebraic expansion of a determinant of order 16. An algorithm has been developed to handle determinants of such large size.

The present paper may also be regarded as an extension of the analysis of Nield and Kuznetsov [7] from the homogeneous case to the heterogeneous case. Further information about heat transfer in bidisperse porous media can be found in the review by Nield and Kuznetsov [8]. Other references to papers on convection in heterogeneous porous media can be found in [4].

At the outset we have made some simplifications. For strong heterogeneity it is convenient to work in terms of heterogeneity of log permeability (or log hydraulic conductivity), but in a study of weak heterogeneity this would just complicate the analysis. For a similar reason, we have not considered random fields. Our assumption of weak heterogeneity allows us to work in terms of approximations involving small quantities, and we work to second order in these.

2. Analysis

We consider a two-dimensional rectangular enclosure (depth d, width L) occupied by a BDPM heated uniformly from below, with applied temperatures T_{l} and T_{u} at the lower boundary ($y^* = 0$) and the upper boundary ($y^* = d$), respectively. This is shown schematically in Fig. 1b. Thus d is the depth of the layer. (The asterisks denote dimensional variables.) The equations of continuity (expressing conservation of mass) for the velocity components in the two phases are

$$\frac{\partial u_f^*}{\partial x^*} + \frac{\partial v_f^*}{\partial y^*} = 0, \tag{1}$$

$$\frac{\partial u_p^*}{\partial x^*} + \frac{\partial v_p^*}{\partial y^*} = 0. \tag{2}$$

We note that in the traditional Darcy formulation the pressure is an intrinsic quantity, i.e. it is the pressure in the fluid. We recognize that in a BDPM the fluid occupies all of the f-phase (the macropore portion) and a fraction of the p-phase (the micropore portion of the porous phase). We denote the volume fraction of the f-phase by ϕ (something that in a regular porous medium would be called the porosity) and the porosity in the p-phase by ε. Thus $1-\phi$ is the volume fraction of the p-phase, and the volume fraction of the BDPM occupied by the fluid is $\phi+(1-\phi)\varepsilon$. The volume average of the temperature over the fluid is

$$T_{\mathrm{F}}^{*}=\frac{\phi T_{f}^{*}+(1-\phi)\varepsilon T_{p}^{*}}{\phi+(1-\phi)\varepsilon}. \tag{3}$$

The drag force (per unit volume) balances the gradient of the excess pressure over hydrostatic. Our basic hypothesis is that in a BDPM the drag is increased by an amount $\zeta(\mathbf{v}_{f}^{*}-\mathbf{v}_{p}^{*})$ for the f-phase and decreased by the same amount for the p-phase.

Within the enclosure the permeability is $K^{*}(x^{*},y^{*})$ and the overall (effective) thermal conductivity is $k^{*}(x^{*},y^{*})$. Accordingly, we write the momentum equations as

$$\frac{\partial p^{*}}{\partial x^{*}}=-\frac{\mu}{K_{f}^{*}}u_{f}^{*}-\zeta(u_{f}^{*}-u_{p}^{*})+\tilde{\mu}\nabla^{2}u_{f}^{*}, \tag{4}$$

$$\frac{\partial p^{*}}{\partial x^{*}}=-\frac{\mu}{K_{p}^{*}}u_{p}^{*}-\zeta(u_{p}^{*}-u_{f}^{*})+\tilde{\mu}\nabla^{2}u_{p}^{*}, \tag{5}$$

$$\frac{\partial p^{*}}{\partial y^{*}}=-\frac{\mu}{K_{f}^{*}}v_{f}^{*}-\zeta(v_{f}^{*}-v_{p}^{*})+\tilde{\mu}\nabla^{2}v_{f}^{*}+\rho_{F}g\hat{\beta}(T_{\mathrm{F}}^{*}-T_{0}), \tag{6}$$

$$\frac{\partial p^{*}}{\partial y^{*}}=-\frac{\mu}{K_{p}^{*}}v_{p}^{*}-\zeta(v_{p}^{*}-v_{f}^{*})+\tilde{\mu}\nabla^{2}v_{p}^{*}+\rho_{F}g\hat{\beta}(T_{\mathrm{F}}^{*}-T_{0}). \tag{7}$$

We have simplified the equations by assuming that $\tilde{\mu}_{f}$ and $\tilde{\mu}_{p}$ are equal, so the subscripts on $\tilde{\mu}$ can be dropped. Here ρ_{F} is the density of the fluid, $\hat{\beta}$ is the volumetric thermal expansion coefficient of the fluid, and T_{0} is a reference temperature.

The thermal energy equations are taken as

$$\phi(\rho c)_{f}\frac{\partial T_{f}^{*}}{\partial t^{*}}+\phi(\rho c)_{f}\mathbf{v}_{f}^{*}\cdot\nabla T_{f}^{*}=\phi k_{f}^{*}\nabla^{2}T_{f}^{*}+h(T_{p}^{*}-T_{f}^{*}), \tag{8}$$

$$(1-\phi)(\rho c)_{p}\frac{\partial T_{p}^{*}}{\partial t^{*}}+(1-\phi)(\rho c)_{p}\mathbf{v}_{p}^{*}\cdot\nabla T_{p}^{*}=(1-\phi)k_{p}^{*}\nabla^{2}T_{p}^{*}+h(T_{f}^{*}-T_{p}^{*}). \tag{9}$$

Here c denotes the specific heat at constant pressure, k^{*} denotes the thermal conductivity, and h is an inter-phase heat transfer coefficient (incorporating the specific area).

In order to simplify the following analysis, on the right-hand side of Eq. (8) the terms involving the partial derivatives of k_{f}^{*} with respect to the spatial coordinates have been dropped. In accordance with the assumption of weak heterogeneity, it is assumed that the variation of k_{f}^{*} over the enclosure is small compared with the mean value of k_{f}^{*}. It can be shown that this approximation has no effect on the results presented in this paper provided that k_{f}^{*} is a linear function of the spatial variables considered separately. A similar approximation involving k_{p}^{*} has been made in Eq. (9). A similar assumption about the variation of the permeability is made below.

We define K_{f0}, K_{p0}, k_{f0} and k_{p0} as the mean values of K_{f}^{*}, K_{p}^{*}, k_{f}^{*} and k_{p}^{*}, respectively, and write

$$\widehat{K}_{f}=K_{f}^{*}/K_{f0},\quad \widehat{K}_{p}=K_{p}^{*}/K_{p0},\quad \hat{k}_{f}=k_{f}^{*}/k_{f0},\quad \hat{k}_{p}=k_{p}^{*}/k_{p0}. \tag{10}$$

We introduce dimensionless variables as follows:

$$(x^{*},y^{*})=d(x,y),\quad t^{*}=\frac{(\rho c)_{f}}{k_{f0}}d^{2}t,\quad p^{*}=\frac{k_{f}\mu}{(\rho c)_{f}K_{f0}}p, \tag{11}$$

$$(u_{f}^{*},v_{f}^{*})=\frac{\phi k_{f0}}{(\rho c)_{f}d}(u_{f},v_{f}),\quad (u_{p}^{*},v_{p}^{*})=\frac{(1-\phi)k_{p0}}{(\rho c)_{p}d}(u_{p},v_{p}), \tag{12}$$

$$T_{f}^{*}=(T_{1}-T_{\mathrm{u}})\theta_{f}+T_{\mathrm{u}},\quad T_{p}^{*}=(T_{1}-T_{\mathrm{u}})\theta_{p}+T_{\mathrm{u}}. \tag{13}$$

We take the reference temperature T_{0} as $T_{1}-T_{\mathrm{u}}$. We also introduce the stream functions ψ_{f} and ψ_{p} defined so that

$$u_{f}=-\frac{\partial\psi_{f}}{\partial y},\quad v_{f}=\frac{\partial\psi_{f}}{\partial x},\quad u_{p}=-\frac{\partial\psi_{p}}{\partial y},\quad v_{p}=\frac{\partial\psi_{p}}{\partial x}. \tag{14}$$

We define a Rayleigh number Ra_{f} and a Darcy number Da_{f} based on properties in the f-phase by

$$Ra_{f}=\frac{\rho_{F}g\hat{\beta}(T_{1}-T_{\mathrm{u}})K_{f0}d}{\mu\phi k_{f0}/(\rho c)_{f}}, \tag{15a}$$

$$Da_{f}=\frac{\tilde{\mu}K_{f0}}{\mu d^{2}}. \tag{15b}$$

Elimination of the pressure from Eqs. (4)–(7), on the assumption that the maximum variation of permeability in the box is a small fraction of the mean permeability so that derivatives of the permeability are small, gives

$$\left[(1+\sigma_{f}\widehat{K}_{f})\nabla^{2}-Da_{f}\widehat{K}_{f}\nabla^{4}\right]\psi_{f}-\beta\sigma_{f}\widehat{K}_{f}\nabla^{2}\psi_{p}=Ra_{f}\widehat{K}_{f}\frac{\partial\theta_{F}}{\partial x}, \tag{16}$$

$$-\sigma_{f}\widehat{K}_{p}\nabla^{2}\psi_{f}+\beta\left[\left(\frac{1}{K_{r}}+\sigma_{f}\widehat{K}_{p}\right)\nabla^{2}-Da_{f}\widehat{K}_{p}\nabla^{4}\right]\psi_{p}=Ra_{f}\widehat{K}_{p}\frac{\partial\theta_{F}}{\partial x}, \tag{17}$$

where

$$\frac{\partial\theta_{F}}{\partial x}=\frac{\varphi\frac{\partial\theta_{f}}{\partial x}+(1-\phi)\varepsilon\frac{\partial\theta_{p}}{\partial x}}{\phi+(1-\phi)\varepsilon}. \tag{18}$$

Here we have introduced the dimensionless parameters

$$\sigma_{f}=\frac{\zeta K_{f0}}{\mu},\quad \beta=\frac{(1-\phi)k_{p0}(\rho c)_{f}}{\phi k_{f0}(\rho c)_{p}},\quad K_{r}=\frac{K_{p0}}{K_{f0}}. \tag{19}$$

Thus σ_{f} is an inter-phase momentum transfer parameter, while β is a modified thermal diffusivity ratio.

Also, the thermal energy equations (8) and (9) become

$$\frac{\partial\theta_{f}}{\partial t}-\frac{\partial\psi_{f}}{\partial y}\frac{\partial\theta_{f}}{\partial x}+\frac{\partial\psi_{f}}{\partial x}\frac{\partial\theta_{f}}{\partial y}=\hat{k}_{f}\nabla^{2}\theta_{f}+H(\theta_{p}-\theta_{f}), \tag{20}$$

$$\alpha\frac{\partial\theta_{p}}{\partial t}-\frac{\partial\psi_{p}}{\partial y}\frac{\partial\theta_{p}}{\partial x}+\frac{\partial\psi_{p}}{\partial x}\frac{\partial\theta_{p}}{\partial y}=\hat{k}_{p}\nabla^{2}\theta_{p}+\gamma H(\theta_{f}-\theta_{p}), \tag{21}$$

where

$$\alpha = \frac{k_{f0}}{k_{p0}}\frac{(\rho c)_p}{(\rho c)_f}, \quad \gamma = \frac{\phi k_{f0}}{(1-\phi)k_{p0}}, \quad H = \frac{hd^2}{\phi k_{f0}}. \tag{22}$$

Thus α is a thermal diffusivity ratio, γ is a modified thermal conductivity ratio, and H is an inter-phase heat transfer parameter.

The conducting state solution is

$$\psi_f = \psi_p = 0, \quad \theta_f = \theta_p = 1 - y. \tag{23}$$

We now perturb this solution and write

$$\psi_f = \Psi_f, \quad \psi_p = \Psi_p, \quad \theta_f = 1 - y + \Theta_f, \quad \theta_p = 1 - y + \Theta_p. \tag{24}$$

We also invoke the principle of exchange of stabilities. This has the effect that the inertial coefficient α drops out of the subsequent equations. Substitution in Eqs. (16)–(21) and linearization gives

$$\left[(1+\sigma_f\widehat{K}_f)\nabla^2 - Da_f\widehat{K}_f\nabla^4\right]\Psi_f - \beta\sigma_f\widehat{K}_f\nabla^2\Psi_p = Ra_f\widehat{K}_f\left[\frac{\phi\frac{\partial\Theta_f}{\partial x} + (1-\phi)\varepsilon\frac{\partial\Theta_p}{\partial x}}{\phi+(1-\phi)\varepsilon}\right], \tag{25}$$

$$-\sigma_f\widehat{K}_p\nabla^2\Psi_f + \beta\left[\left(\frac{1}{K_r}+\sigma_f\widehat{K}_p\right)\nabla^2 - Da_f\widehat{K}_p\nabla^4\right]\Psi_p = Ra_f\widehat{K}_p\left[\frac{\phi\frac{\partial\Theta_f}{\partial x} + (1-\phi)\varepsilon\frac{\partial\Theta_p}{\partial x}}{\phi+(1-\phi)\varepsilon}\right]. \tag{26}$$

$$\frac{\partial\Theta_f}{\partial t} = \hat{k}_f\nabla^2\Theta_f + \frac{\partial\Psi_f}{\partial x} + H(\Theta_p - \Theta_f), \tag{27}$$

$$\alpha\frac{\partial\Theta_p}{\partial t} = \hat{k}_p\nabla^2\Theta_p + \frac{\partial\Psi_p}{\partial x} + \gamma H(\Theta_f - \Theta_p). \tag{28}$$

As a final scaling for mathematical convenience, one can transform the rectangular domain to a square by means of the transformation

$$x = A\hat{x}, \quad y = \hat{y}, \tag{29}$$

where A is the depth-to-width aspect ratio

$$A = d/L. \tag{30}$$

The differential equations take the matrix form

$$\mathbf{LY} = 0, \tag{31}$$

where

$$\mathbf{Y} = (\Psi_f, \Psi_p, \Theta_f, \Theta_p)^{\mathrm{T}}, \tag{32}$$

$$L_{11} = (1+\sigma_f\widehat{K}_f)\left(A^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - Da_f\widehat{K}_f\left(A^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2,$$

$$L_{12} = -\beta\sigma_f\widehat{K}_f\left(A^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right),$$

$$L_{13} = -\frac{\phi A Ra_f\widehat{K}_f}{\phi+(1-\phi)\varepsilon}\frac{\partial}{\partial x},$$

$$L_{14} = -\frac{(1-\phi)\varepsilon A Ra_f\widehat{K}_f}{\phi+(1-\phi)\varepsilon}\frac{\partial}{\partial x},$$

$$L_{21} = -\sigma_f\widehat{K}_p\left(A^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right),$$

$$L_{22} = \beta\left[\left(\frac{1}{K_r}+\sigma_f\widehat{K}_p\right)\left(A^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - Da_f\widehat{K}_p\left(A^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\right],$$

$$L_{23} = -\frac{\phi A Ra_f\widehat{K}_p}{\phi+(1-\phi)\varepsilon}\frac{\partial}{\partial x},$$

$$L_{24} = -\frac{(1-\phi)\varepsilon A Ra_f\widehat{K}_p}{\phi+(1-\phi)\varepsilon}\frac{\partial}{\partial x},$$

$$L_{31} = L_{42} = A\frac{\partial}{\partial x}, \quad L_{32} = L_{41} = 0,$$

$$L_{33} = \hat{k}_f\left(A^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - H, \quad L_{34} = H,$$

$$L_{43} = \gamma H, \quad L_{44} = \hat{k}_p\left(A^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - \gamma H. \tag{33}$$

For conducting stress-free top and bottom boundaries and insulating stress-free side walls, the boundary conditions are

$$\begin{aligned}&\Psi_f = \Psi_p = \frac{\partial^2\Psi_f}{\partial x^2} = \frac{\partial^2\Psi_p}{\partial x^2} = \frac{\partial\Theta_f}{\partial x} = \frac{\partial\Theta_p}{\partial x} = 0\\ &\quad \text{at } x = 0 \text{ and at } x = 1,\\ &\Psi_f = \Psi_p = \frac{\partial^2\Psi_f}{\partial y^2} = \frac{\partial^2\Psi_p}{\partial y^2} = \Theta_f = \Theta_p = 0\\ &\quad \text{at } y = 0 \text{ and at } y = 1.\end{aligned} \tag{34}$$

This set of boundary conditions is satisfied by functions of the form

$$\Psi_{mn} = \sin m\pi x \sin n\pi y, \quad m, n = 1, 2, 3, \ldots, \tag{35}$$

$$\Theta_{pq} = \cos p\pi x \sin q\pi y, \quad p, q = 1, 2, 3, \ldots \tag{36}$$

We can take this set of functions (that are exact eigenfunctions for the homogeneous case) as trial functions for an approximate solution of the heterogeneous case. For example, working at second order, we can try

$$\begin{aligned}\Psi_f &= A_{11}\Psi_{11} + A_{12}\Psi_{12} + A_{21}\Psi_{21} + A_{22}\Psi_{22},\\ \Psi_P &= B_{11}\Psi_{11} + B_{12}\Psi_{12} + B_{21}\Psi_{21} + B_{22}\Psi_{22},\\ \Theta_f &= C_{11}\Theta_{11} + C_{12}\Theta_{12} + C_{21}\Theta_{21} + C_{22}\Theta_{22},\\ \Theta_p &= D_{11}\Theta_{11} + D_{12}\Theta_{12} + D_{21}\Theta_{21} + D_{22}\Theta_{22}.\end{aligned} \tag{37}$$

Let R_1, R_2, R_3, R_4, be the residuals when the four expressions (37) are substituted in the left-hand sides of the four differential equations involved in Eq. (31). In turn, R_1 and R_2 are made orthogonal to Ψ_{11}, Ψ_{12}, Ψ_{21}, Ψ_{22} and R_3 and R_4 are made orthogonal to Θ_{11}, Θ_{12}, Θ_{21}, Θ_{22}.

The result is 16 linear homogeneous equations in the 16 constants $A_{11},\ldots,D_{22}$ whose solution requires the vanishing of the determinant of coefficients, namely

$$\det \mathbf{M} = 0, \tag{38}$$

where $\mathbf{M}$ is a 16 by 16 matrix.

In the general case, the integrals in the matrix elements can be obtained by quadrature. The eigenvalue equation, Eq. (38) can then be solved to give the critical Rayleigh number Ra.

We consider a quartered square in which each slowly varying quantity is approximated by a piecewise-constant distribution. The mean value of the quantity is approximated by its value at centre of the main square:

$$\bar{f} = f(0.5, 0.5). \tag{39}$$

In each quarter, the function is approximated by its value at the centre of that quarter, and a truncated Taylor series expansion is used to approximate this quantity. For example, in the region $1/2 < x < 1,\ 1/2 < y < 1, f(x,y)$ is approximated by $f(0.75, 0.75)$ and then by

$$f(0.5,0.5) + 0.25 f_x(0.5,0.5) + 0.25 f_y(0.5,0.5).$$

Hence we consider the case

$$\begin{aligned}
&\widehat{K}_f(x,y) = 1 - \delta_{f\mathrm{H}} - \delta_{f\mathrm{V}}, \hat{k}_f(x,y) = 1 - \varepsilon_{f\mathrm{H}} - \varepsilon_{f\mathrm{V}},\\
&\quad \text{for } 0 < x < 1/2,\ 0 < y < 1/2,\\
&\widehat{K}_f(x,y) = 1 + \delta_{f\mathrm{H}} - \delta_{f\mathrm{V}}, \hat{k}_f(x,y) = 1 + \varepsilon_{f\mathrm{H}} - \varepsilon_{f\mathrm{V}},\\
&\quad \text{for } 1/2 < x < 1,\ 0 < y < 1/2,\\
&\widehat{K}_f(x,y) = 1 - \delta_{f\mathrm{H}} + \delta_{f\mathrm{V}}, \hat{k}_f(x,y) = 1 - \varepsilon_{f\mathrm{H}} + \varepsilon_{f\mathrm{V}},\\
&\quad \text{for } 0 < x < 1/2,\ 1/2 < y < 1,\\
&\widehat{K}_f(x,y) = 1 + \delta_{f\mathrm{H}} + \delta_{f\mathrm{V}}, \hat{k}_f(x,y) = 1 + \varepsilon_{f\mathrm{H}} + \varepsilon_{f\mathrm{V}},\\
&\quad \text{for } 1/2 < x < 1,\ 1/2 < y < 1,\\
&\widehat{K}_p(x,y) = 1 - \delta_{p\mathrm{H}} - \delta_{p\mathrm{V}}, \hat{k}_p(x,y) = 1 - \varepsilon_{p\mathrm{H}} - \varepsilon_{p\mathrm{V}},\\
&\quad \text{for } 0 < x < 1/2,\ 0 < y < 1/2,\\
&\widehat{K}_p(x,y) = 1 + \delta_{p\mathrm{H}} - \delta_{p\mathrm{V}}, \hat{k}_p(x,y) = 1 + \varepsilon_{p\mathrm{H}} - \varepsilon_{p\mathrm{V}},\\
&\quad \text{for } 1/2 < x < 1,\ 0 < y < 1/2,\\
&\widehat{K}_p(x,y) = 1 - \delta_{p\mathrm{H}} + \delta_{p\mathrm{V}}, \hat{k}_p(x,y) = 1 - \varepsilon_{p\mathrm{H}} + \varepsilon_{p\mathrm{V}},\\
&\quad \text{for } 0 < x < 1/2,\ 1/2 < y < 1,\\
&\widehat{K}_p(x,y) = 1 + \delta_{p\mathrm{H}} + \delta_{p\mathrm{V}}, \hat{k}_p(x,y) = 1 + \varepsilon_{p\mathrm{H}} + \varepsilon_{p\mathrm{V}},\\
&\quad \text{for } 1/2 < x < 1,\ 1/2 < y < 1,
\end{aligned} \tag{40}$$

where, for example,

$$\delta_{f\mathrm{H}} = \frac{1}{4}\left[\frac{\partial\widehat{K}_f/\partial x}{\widehat{K}_f}\right]_{x=1/2,y=1/2}, \quad \delta_{f\mathrm{V}} = \frac{1}{4}\left[\frac{\partial\widehat{K}_f/\partial y}{\widehat{K}_f}\right]_{x=1/2,y=1/2}. \tag{41}$$

We introduce the shorthand notation

$$\begin{aligned}&[\Delta_{f\mathrm{H}}, \Delta_{f\mathrm{V}}, \Delta_{p\mathrm{H}}, \Delta_{p\mathrm{V}}, E_{f\mathrm{H}}, E_{f\mathrm{V}}, E_{p\mathrm{H}}, E_{p\mathrm{V}}]\\ &\quad = (8/3\pi)[\delta_{f\mathrm{H}}, \delta_{f\mathrm{V}}, \delta_{p\mathrm{H}}, \delta_{p\mathrm{V}}, \varepsilon_{f\mathrm{H}}/2, \varepsilon_{f\mathrm{V}}, \varepsilon_{p\mathrm{H}}/2, \varepsilon_{p\mathrm{V}}].\end{aligned} \tag{42}$$

An analytical expansion of a general determinant of order 16 involves 2×10^{13} terms and so is obviously impractical. However, the determinant of a quasi-diagonalized matrix $\mathbf{M}$ (one in which all the elements off the principal diagonal are small) can be approximated, to second order in small quantities, as follows.

Define the trace of $\mathbf{M}$ as

$$\mathrm{Tr} = M(1,1)M(2,2)\cdots M(16,16).$$

Initialize $D = \mathrm{Tr}$.

For $i = 1,\ldots,15;\ j = i+1,\ldots,16$

$$D = D - \frac{M(i,j)M(j,i)}{M(i,i)M(j,j)}\mathrm{Tr}. \tag{43}$$

The final value of D gives $\det\mathbf{M}$.

A proof of the validity of the algorithm is based on an expansion according to the minors of the last two columns (or rows) and induction.

This expression given by the algorithm is already approximate to second order in small quantities occupying the off-diagonal elements.

The details of the evaluation of the matrix elements are omitted here for simplicity. The evaluation follows the pattern in [4].

Using elementary row and column transformations, the present determinant can be put in diagonal form as follows.

Define

$$\psi_f = \frac{\phi + (1-\phi)\varepsilon}{\phi},$$

$$\psi_p = \frac{\phi + (1-\phi)\varepsilon}{(1-\phi)\varepsilon},$$

$$Z_{mn} = (m^2A^2 + n^2)\pi^2,$$

$$B_{mn} = \frac{Da_f Z_{mn} + \sigma_f}{Da_f Z_{mn} + 1 + \sigma_f},$$

$$C_{mn} = \frac{Da_f Z_{mn} + K_r^{-1} + \sigma_f}{\sigma_f},$$

$$D_{mn} = \frac{Da_f Z_{mn} + \sigma_f}{Da_f Z_{mn} + K_r^{-1} + \sigma_f},$$

$$E_{mn} = \frac{\sigma_f}{Da_f Z_{mn} + 1 + \sigma_f},$$

$$F_{mn} = \frac{Z_{mn} + H}{m\pi A},$$
$$G_{mn} = \frac{Z_{mn}}{Z_{mn} + H},$$
$$H_{mn} = \frac{m\pi A}{Da_f Z_{mn}^2 + (1+\sigma_f)Z_{mn}},$$
$$I_{mn} = \frac{m\pi A}{\beta\sigma_f Z_{mn}},$$
$$K_{mn} = \frac{1}{m\pi A},$$
$$L_{mn} = \frac{Z_{mn} + \gamma H}{m\pi A},$$
$$M_{mn} = \frac{Z_{mn}}{Z_{mn} + \gamma H},$$
$$P_{mn} = \frac{1}{m\pi A}[(Z_{mn} + H)\psi_f + H\psi_p],$$
$$Q_{mn} = \frac{1}{m\pi A}[(Z_{mn} + \gamma H)\psi_p + \gamma H\psi_f],$$
$$R_{mn} = \frac{Da_f^2 Z_{mn}^2 + (1 + K_r^{-1} + 2\sigma_f)Da_f Z_{mn} + K_r^{-1} + (K_r^{-1} + 1)\sigma_f}{\sigma_f(Da_f Z_{mn} + 1 + 2\sigma_f)},$$
$$S_{mn} = \frac{m\pi A}{\sigma_f Z_{mn}}\left[\frac{Da_f Z_{mn} + K_r^{-1} + 2\sigma_f}{Da_f Z_{mn} + 1 + 2\sigma_f} + \frac{(Z_{mn} + H)\psi_f + H\psi_p}{\beta[(Z_{mn} + \gamma H)\psi_p + \gamma H\psi_f]}\right],$$
$$T_{mn} = \frac{Z_{mn}(Z_{mn} + H + \gamma H)\psi_f\psi_p}{m\pi A[(Z_{mn} + \gamma H)\psi_p + \gamma H\psi_f]},$$
$$V_{mn} = \frac{Da_f Z_{mn}}{\sigma_f} + \frac{1 - K_r^{-1}}{Da_f Z_{mn} + 1 + 2\sigma_f},$$
$$W_{mn} = 1 - \frac{(Da_f Z_{mn} + K_r^{-1} + 2\sigma_f)(Da_f Z_{mn} + \sigma_f)}{\sigma_f(Da_f Z_{mn} + 1 + 2\sigma_f)},$$
$$Y_{mn} = \frac{Z_{mn}}{m\pi A},$$
$$AA_{mn} = \frac{I_{mn}Y_{mn}}{Q_{mn}} + \frac{S_{mn}K_{mn}Y_{mn}}{T_{mn}Q_{mn}}\gamma H\psi_f\psi_p,$$
$$BB_{mn} = \frac{K_{mn}Y_{mn}}{Q_{mn}},$$
$$CC_{mn} = \frac{Ra}{1 + E_{mn}}\left[\frac{H_{mn}}{T_{mn}}Ra + E_{mn}\right],$$
$$DD_{mn} = \frac{Ra}{1 + E_{mn}}\left[\frac{H_{mn}}{T_{mn}}Ra + E_{mn}\right] - Ra,$$
$$EE_{mn} = \frac{S_{mn}}{T_{mn}}Ra - R_{mn},$$
$$FF_{mn} = \frac{H_{mn}}{T_{mn}}Ra + E_{mn},$$
$$GG_{mn} = \frac{S_{mn}}{T_{mn}}Ra + W_{mn},$$
$$HH_{mn} = -\frac{S_{mn}}{T_{mn}}Ra + V_{mn},$$
$$JJ_{mn} = \frac{H_{mn}}{1 + E_{mn}}Ra,$$
$$KK_{mn} = Y_{mn}\psi_f\left(\frac{K_{mn}}{Q_{mn}}\gamma H\psi_f - 1\right),$$
$$LL_{mn} = \left(\frac{S_{mn}}{T_{mn}}\left[1 - \frac{K_{mn}}{Q_{mn}}\gamma H\right] - \frac{I_{mn}}{Q_{mn}}\right)Y_{mn}\psi_f,$$
$$\alpha_{mnpq} = \frac{FF_{mn}B_{pq}}{1 + E_{mn}},$$
$$\beta_{mnpq} = \frac{FF_{mn}E_{pq}}{1 + E_{mn}},$$
$$\gamma_{mnpq} = \frac{H_{mn}B_{pq}}{1 + E_{mn}},$$
$$\delta_{mnpq} = \frac{H_{mn}E_{pq}}{1 + E_{mn}},$$
$$\eta_{mnpq} = \frac{K_{mn}Y_{mn}P_{pq}}{Q_{mn}Q_{pq}}\gamma H\psi_f\psi_p,$$
$$\zeta_{mnpq} = \frac{FF_{mn}}{1 + E_{mn}}\left[W_{pq} + \frac{S_{pq}}{T_{pq}}Ra\right],$$
$$\theta_{mnpq} = \left[\frac{FF_{mn}}{1 + E_{mn}} - 1\right]\left[V_{pq} - \frac{S_{pq}}{T_{pq}}Ra\right],$$
$$\lambda_{mnpq} = \frac{H_{mn}}{1 + E_{mn}}\left[W_{pq} + \frac{S_{pq}}{T_{pq}}Ra\right],$$
$$\mu_{mnpq} = \frac{H_{mn}}{1 + E_{mn}}\left[-V_{pq} + \frac{S_{pq}}{T_{pq}}Ra\right],$$
$$\nu_{mnpq} = \frac{P_{mn}I_{pq}Y_{pq}}{Q_{mn}Q_{pq}}\psi_p + \frac{\gamma HP_{mn}S_{pq}K_{pq}Y_{pq}}{Q_{mn}Q_{pq}T_{pq}}\psi_f\psi_p,$$
$$\pi_{mnpq} = \frac{Ra}{T_{mn}}\left(\frac{S_{pq}Y_{pq}}{T_{pq}} - \frac{I_{pq}Y_{pq}}{Q_{pq}}\psi_f - \frac{S_{pq}K_{pq}Y_{pq}}{T_{pq}Q_{pq}}\gamma H\psi_f^2\right),$$
$$\rho_{mnpq} = \frac{RaP_{mn}}{T_{mn}Q_{mn}}\left(\frac{I_{pq}Y_{pq}}{Q_{pq}}\psi_f + \frac{S_{pq}K_{pq}Y_{pq}}{T_{pq}Q_{pq}}\gamma H\psi_f\psi_p\right),$$
$$\sigma_{mnpq} = \frac{RaY_{pq}}{T_{mn}}\psi_f,$$
$$\tau_{mnpq} = \frac{RaP_{mn}Y_{pq}}{T_{mn}Q_{mn}}\psi_p,$$
$$\upsilon_{mnpq} = \frac{Ra}{T_{mn}}\psi_f\left(Y_{pq} - \frac{K_{pq}Y_{pq}}{Q_{pq}}\gamma H\psi_f\right),$$
$$\varphi_{mnpq} = \frac{RaP_{mn}K_{pq}Y_{pq}}{T_{mn}Q_{mn}Q_{pq}}\gamma H\psi_f\psi_p,$$
$$\omega_{mnpq} = \frac{Y_{mn}P_{pq}}{Q_{pq}}. \tag{44}$$

Then the diagonalized matrix takes the form

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} & \mathbf{M}_{14} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} & \mathbf{M}_{24} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} & \mathbf{M}_{34} \\ \mathbf{M}_{41} & \mathbf{M}_{42} & \mathbf{M}_{43} & \mathbf{M}_{44} \end{bmatrix}, \tag{45}$$

where

$$\mathbf{M}_{11}=\begin{bmatrix} 1+E_{11} & -B_{12}\Delta_{f\mathrm{V}}-E_{12}\Delta_{p\mathrm{V}} & -B_{21}\Delta_{f\mathrm{H}}-E_{21}\Delta_{p\mathrm{H}} & 0 \\ -B_{11}\Delta_{f\mathrm{V}}-E_{11}\Delta_{p\mathrm{V}} & 1+E_{12} & 0 & -B_{22}\Delta_{f\mathrm{H}}-E_{22}\Delta_{p\mathrm{H}} \\ -B_{11}\Delta_{f\mathrm{H}}-E_{11}\Delta_{p\mathrm{H}} & 0 & 1+E_{21} & -B_{22}\Delta_{f\mathrm{V}}-E_{22}\Delta_{p\mathrm{V}} \\ 0 & -B_{12}\Delta_{f\mathrm{H}}-E_{12}\Delta_{p\mathrm{H}} & -B_{21}\Delta_{f\mathrm{V}}-E_{21}\Delta_{p\mathrm{V}} & 1+E_{22} \end{bmatrix},$$

$$\mathbf{M}_{12}=\begin{bmatrix} 0 & GG_{12}\Delta_{f\mathrm{V}}+HH_{12}\Delta_{p\mathrm{V}} & GG_{21}\Delta_{HV}+HH_{21}\Delta_{p\mathrm{H}} & 0 \\ GG_{11}\Delta_{f\mathrm{V}}+HH_{11}\Delta_{p\mathrm{V}} & 0 & 0 & GG_{22}\Delta_{f\mathrm{H}}+HH_{22}\Delta_{p\mathrm{H}} \\ GG_{11}\Delta_{f\mathrm{H}}+HH_{11}\Delta_{p\mathrm{H}} & 0 & 0 & GG_{22}\Delta_{f\mathrm{V}}+HH_{22}\Delta_{p\mathrm{V}} \\ 0 & GG_{12}\Delta_{f\mathrm{H}}+HH_{12}\Delta_{p\mathrm{H}} & GG_{21}\Delta_{f\mathrm{V}}+HH_{21}\Delta_{p\mathrm{V}} & 0 \end{bmatrix},$$

$$\mathbf{M}_{13}=\begin{bmatrix} 0 & Ra(\Delta_{p\mathrm{V}}-\Delta_{f\mathrm{V}}) & Ra(\Delta_{p\mathrm{H}}-\Delta_{f\mathrm{H}}) & 0 \\ Ra(\Delta_{p\mathrm{V}}-\Delta_{f\mathrm{V}}) & 0 & 0 & Ra(\Delta_{p\mathrm{H}}-\Delta_{f\mathrm{H}}) \\ Ra(\Delta_{p\mathrm{H}}-\Delta_{f\mathrm{H}}) & 0 & 0 & Ra(\Delta_{p\mathrm{V}}-\Delta_{f\mathrm{V}}) \\ 0 & Ra(\Delta_{p\mathrm{H}}-\Delta_{f\mathrm{H}}) & Ra(\Delta_{p\mathrm{V}}-\Delta_{f\mathrm{V}}) & 0 \end{bmatrix},\quad \mathbf{M}_{14}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{M}_{21}=\begin{bmatrix} 0 & -\alpha_{1112}\Delta_{f\mathrm{V}}+(E_{12}-\beta_{1112})\Delta_{p\mathrm{V}} & -\alpha_{1121}\Delta_{f\mathrm{H}}+(E_{21}-\beta_{1121})\Delta_{p\mathrm{H}} & 0 \\ -\alpha_{1211}\Delta_{f\mathrm{V}}+(E_{11}-\beta_{1211})\Delta_{p\mathrm{V}} & 0 & 0 & -\alpha_{1222}\Delta_{f\mathrm{H}}+(E_{22}-\beta_{1222})\Delta_{p\mathrm{H}} \\ -\alpha_{2111}\Delta_{f\mathrm{H}}+(E_{11}-\beta_{2111})\Delta_{p\mathrm{H}} & 0 & 0 & -\alpha_{2122}\Delta_{f\mathrm{V}}+(E_{22}-\beta_{2122})\Delta_{p\mathrm{V}} \\ 0 & -\alpha_{2212}\Delta_{f\mathrm{H}}+(E_{12}-\beta_{2212})\Delta_{p\mathrm{H}} & -\alpha_{2221}\Delta_{f\mathrm{V}}+(E_{21}-\beta_{2221})\Delta_{p\mathrm{V}} & 0 \end{bmatrix}$$

$$\mathbf{M}_{22}=[\mathbf{M}_{221},\mathbf{M}_{222},\mathbf{M}_{223},\mathbf{M}_{224}],$$

where

$$\mathbf{M}_{221}=\begin{bmatrix} -EE_{11} \\ \zeta_{1211}\Delta_{f\mathrm{V}}+\theta_{1211}\Delta_{p\mathrm{V}}-\pi_{1211}E_{f\mathrm{V}}-\rho_{1211}E_{p\mathrm{V}} \\ \zeta_{2111}\Delta_{f\mathrm{H}}+\theta_{2111}\Delta_{p\mathrm{H}}-\pi_{2111}E_{f\mathrm{H}}-\rho_{2111}E_{p\mathrm{H}} \\ 0 \end{bmatrix},$$

$$\mathbf{M}_{222}=\begin{bmatrix} \zeta_{1112}\Delta_{f\mathrm{V}}+\theta_{1112}\Delta_{p\mathrm{V}}-\pi_{1112}E_{f\mathrm{V}}-\rho_{1112}E_{p\mathrm{V}} \\ -EE_{12} \\ 0 \\ \zeta_{2212}\Delta_{f\mathrm{H}}+\theta_{2212}\Delta_{p\mathrm{H}}-\pi_{2212}E_{f\mathrm{H}}-\rho_{2212}E_{p\mathrm{H}} \end{bmatrix},$$

$$\mathbf{M}_{223}=\begin{bmatrix} \zeta_{1121}\Delta_{f\mathrm{H}}+\theta_{1121}\Delta_{p\mathrm{H}}-\pi_{1121}E_{f\mathrm{H}}-\rho_{1121}E_{p\mathrm{H}} \\ 0 \\ -EE_{21} \\ \zeta_{2221}\Delta_{f\mathrm{V}}+\theta_{2221}\Delta_{p\mathrm{V}}-\pi_{2221}E_{f\mathrm{V}}-\rho_{2221}E_{p\mathrm{V}} \end{bmatrix},$$

$$\mathbf{M}_{224}=\begin{bmatrix} 0 \\ \zeta_{1222}\Delta_{f\mathrm{H}}+\theta_{1222}\Delta_{p\mathrm{H}}-\pi_{1222}E_{f\mathrm{H}}-\rho_{1222}E_{p\mathrm{H}} \\ \zeta_{2122}\Delta_{f\mathrm{V}}+\theta_{2122}\Delta_{p\mathrm{V}}-\pi_{2122}E_{f\mathrm{V}}-\rho_{2122}E_{p\mathrm{V}} \\ -EE_{22} \end{bmatrix},$$

$$\mathbf{M}_{23}=[\mathbf{M}_{231},\mathbf{M}_{232},\mathbf{M}_{233},\mathbf{M}_{234}],$$

$$\mathbf{M}_{231}=\begin{bmatrix} 0 \\ -CC_{12}\Delta_{f\mathrm{V}}+DD_{12}\Delta_{p\mathrm{V}}+\upsilon_{1211}E_{f\mathrm{V}}+\varphi_{1211}E_{p\mathrm{V}} \\ -CC_{21}\Delta_{f\mathrm{H}}+DD_{21}\Delta_{p\mathrm{H}}+\upsilon_{2111}E_{f\mathrm{H}}+\varphi_{2111}E_{p\mathrm{H}} \\ 0 \end{bmatrix},$$

$$\mathbf{M}_{232}=\begin{bmatrix} -CC_{11}\Delta_{f\mathrm{V}}+DD_{11}\Delta_{p\mathrm{V}}+\upsilon_{1112}E_{f\mathrm{V}}+\varphi_{1112}E_{p\mathrm{V}} \\ 0 \\ 0 \\ -CC_{22}\Delta_{f\mathrm{H}}+DD_{22}\Delta_{p\mathrm{H}}+\upsilon_{2212}E_{f\mathrm{H}}+\varphi_{2212}E_{p\mathrm{H}} \end{bmatrix},$$

$$\mathbf{M}_{233}=\begin{bmatrix} -CC_{11}\Delta_{f\mathrm{H}}+DD_{11}\Delta_{p\mathrm{H}}+\upsilon_{1121}E_{f\mathrm{H}}+\varphi_{11121}E_{p\mathrm{H}} \\ 0 \\ 0 \\ -CC_{22}\Delta_{f\mathrm{V}}+DD_{22}\Delta_{p\mathrm{V}}+\upsilon_{2221}E_{f\mathrm{V}}+\varphi_{2221}E_{p\mathrm{V}} \end{bmatrix},$$

$$\mathbf{M}_{234}=\begin{bmatrix} 0 \\ -CC_{12}\Delta_{f\mathrm{H}}+DD_{12}\Delta_{p\mathrm{H}}+\upsilon_{1222}E_{f\mathrm{H}}+\varphi_{1222}E_{p\mathrm{H}} \\ -CC_{21}\Delta_{f\mathrm{V}}+DD_{21}\Delta_{p\mathrm{V}}+\upsilon_{2122}E_{f\mathrm{V}}+\varphi_{2122}E_{p\mathrm{V}} \\ 0 \end{bmatrix},$$

$$\mathbf{M}_{24} = \begin{bmatrix} 0 & -\sigma_{1112}E_{f\mathrm{V}} + \tau_{1112}E_{p\mathrm{V}} & -\sigma_{1121}E_{f\mathrm{H}} + \tau_{1121}E_{p\mathrm{H}} & 0 \\ -\sigma_{1211}E_{f\mathrm{V}} + \tau_{1211}E_{p\mathrm{V}} & 0 & 0 & -\sigma_{1222}E_{f\mathrm{H}} + \tau_{1222}E_{p\mathrm{H}} \\ -\sigma_{2111}E_{f\mathrm{H}} + \tau_{2111}E_{p\mathrm{H}} & 0 & 0 & -\sigma_{2122}E_{f\mathrm{V}} + \tau_{2122}E_{p\mathrm{V}} \\ 0 & -\sigma_{2212}E_{f\mathrm{H}} + \tau_{2212}E_{p\mathrm{H}} & -\sigma_{2221}E_{f\mathrm{V}} + \tau_{2221}E_{p\mathrm{V}} & 0 \end{bmatrix},$$

$$\mathbf{M}_{31} = \begin{bmatrix} 0 & \gamma_{1112}\Delta_{f\mathrm{V}} + \delta_{1112}\Delta_{p\mathrm{V}} & \gamma_{1121}\Delta_{f\mathrm{H}} + \delta_{1121}\Delta_{p\mathrm{H}} & 0 \\ \gamma_{1211}\Delta_{f\mathrm{V}} + \delta_{1211}\Delta_{p\mathrm{V}} & 0 & 0 & \gamma_{1222}\Delta_{f\mathrm{H}} + \delta_{1222}\Delta_{p\mathrm{H}} \\ \gamma_{2111}\Delta_{f\mathrm{H}} + \delta_{2111}\Delta_{p\mathrm{H}} & 0 & 0 & \gamma_{2122}\Delta_{f\mathrm{V}} + \delta_{2122}\Delta_{p\mathrm{V}} \\ 0 & \gamma_{2212}\Delta_{f\mathrm{H}} + \delta_{2212}\Delta_{p\mathrm{H}} & \gamma_{2221}\Delta_{f\mathrm{V}} + \delta_{2221}\Delta_{p\mathrm{V}} & 0 \end{bmatrix},$$

$$\mathbf{M}_{32} = [\mathbf{M}_{321}, \mathbf{M}_{322}, \mathbf{M}_{323}, \mathbf{M}_{324}],$$

where

$$\mathbf{M}_{321} = \begin{bmatrix} 0 \\ -\lambda_{1211}\Delta_{f\mathrm{V}} + \mu_{1211}\Delta_{p\mathrm{V}} + LL_{11}E_{f\mathrm{V}} + v_{1211}E_{p\mathrm{V}} \\ -\lambda_{2111}\Delta_{f\mathrm{H}} + \mu_{2111}\Delta_{p\mathrm{H}} + LL_{11}E_{f\mathrm{H}} + v_{2111}E_{p\mathrm{H}} \\ 0 \end{bmatrix},$$

$$\mathbf{M}_{322} = \begin{bmatrix} -\lambda_{1112}\Delta_{f\mathrm{V}} + \mu_{1112}\Delta_{p\mathrm{V}} + LL_{12}E_{f\mathrm{V}} + v_{1112}E_{p\mathrm{V}} \\ 0 \\ 0 \\ -\lambda_{2212}\Delta_{f\mathrm{H}} + \mu_{2212}\Delta_{p\mathrm{H}} + LL_{12}E_{f\mathrm{H}} + v_{2212}E_{p\mathrm{H}} \end{bmatrix},$$

$$\mathbf{M}_{323} = \begin{bmatrix} -\lambda_{1121}\Delta_{f\mathrm{H}} + \mu_{1121}\Delta_{p\mathrm{H}} + LL_{21}E_{f\mathrm{H}} + v_{1121}E_{p\mathrm{H}} \\ 0 \\ 0 \\ -\lambda_{2221}\Delta_{f\mathrm{V}} + \mu_{2221}\Delta_{p\mathrm{V}} + LL_{21}E_{f\mathrm{V}} + v_{2221}E_{p\mathrm{V}} \end{bmatrix},$$

$$\mathbf{M}_{324} = \begin{bmatrix} 0 \\ -\lambda_{1222}\Delta_{f\mathrm{H}} + \mu_{1222}\Delta_{p\mathrm{H}} + LL_{22}E_{f\mathrm{H}} + v_{1222}E_{p\mathrm{H}} \\ -\lambda_{2122}\Delta_{f\mathrm{V}} + \mu_{2122}\Delta_{p\mathrm{V}} + LL_{22}E_{f\mathrm{V}} + v_{2122}E_{p\mathrm{V}} \\ 0 \end{bmatrix},$$

$$\mathbf{M}_{33} = [\mathbf{M}_{331}, \mathbf{M}_{332}, \mathbf{M}_{333}, \mathbf{M}_{334}],$$

where

$$\mathbf{M}_{331} = \begin{bmatrix} T_{11} \\ JJ_{12}\Delta_{f\mathrm{V}} - JJ_{12}\Delta_{p\mathrm{V}} + KK_{11}E_{f\mathrm{V}} - \eta_{1112}E_{p\mathrm{V}} \\ JJ_{21}\Delta_{f\mathrm{H}} - JJ_{12}\Delta_{p\mathrm{H}} + KK_{11}E_{f\mathrm{H}} - \eta_{1121}E_{p\mathrm{H}} \\ 0 \end{bmatrix},$$

$$\mathbf{M}_{332} = \begin{bmatrix} J_{11}\Delta_{f\mathrm{V}} - JJ_{11}\Delta_{p\mathrm{V}} + KK_{12}E_{f\mathrm{V}} - \eta_{1211}E_{p\mathrm{V}} \\ T_{12} \\ 0 \\ JJ_{22}\Delta_{f\mathrm{H}} - JJ_{22}\Delta_{p\mathrm{H}} + KK_{12}E_{f\mathrm{H}} - \eta_{1222}E_{p\mathrm{H}} \end{bmatrix},$$

$$\mathbf{M}_{333} = \begin{bmatrix} JJ_{11}\Delta_{f\mathrm{H}} - JJ_{11}\Delta_{p\mathrm{H}} + KK_{21}E_{f\mathrm{H}} - \eta_{2111}E_{p\mathrm{H}} \\ 0 \\ T_{21} \\ JJ_{22}\Delta_{f\mathrm{V}} - JJ_{22}\Delta_{p\mathrm{V}} + KK_{21}E_{f\mathrm{V}} - \eta_{2122}E_{p\mathrm{V}} \end{bmatrix},$$

$$\mathbf{M}_{334} = \begin{bmatrix} 0 \\ JJ_{12}\Delta_{f\mathrm{H}} - JJ_{12}\Delta_{p\mathrm{H}} + KK_{22}E_{f\mathrm{H}} - \eta_{2212}E_{p\mathrm{H}} \\ JJ_{21}\Delta_{f\mathrm{V}} - JJ_{21}\Delta_{p\mathrm{V}} + KK_{22}E_{f\mathrm{V}} - \eta_{2221}E_{p\mathrm{V}} \\ T_{22} \end{bmatrix},$$

$$\mathbf{M}_{34} = \begin{bmatrix} 0 & Y_{12}\psi_f E_{f\mathrm{V}} - \omega_{1211}\psi_p E_{p\mathrm{V}} & Y_{21}\psi_f E_{f\mathrm{H}} - \omega_{2111}\psi_p E_{p\mathrm{H}} & 0 \\ Y_{11}\psi_f E_{f\mathrm{V}} - \omega_{1112}\psi_p E_{p\mathrm{V}} & 0 & 0 & Y_{22}\psi_f E_{f\mathrm{H}} - \omega_{2212}\psi_p E_{p\mathrm{H}} \\ Y_{11}\psi_f E_{f\mathrm{H}} - \omega_{1121}\psi_p E_{p\mathrm{H}} & 0 & 0 & Y_{22}\psi_f E_{f\mathrm{V}} - \omega_{2221}\psi_p E_{p\mathrm{V}} \\ 0 & Y_{12}\psi_f E_{f\mathrm{H}} - \omega_{1222}\psi_p E_{p\mathrm{H}} & Y_{21}\psi_f E_{f\mathrm{V}} - \omega_{2122}\psi_p E_{p\mathrm{V}} & 0 \end{bmatrix}$$

$$\mathbf{M}_{41} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{M}_{42} = \begin{bmatrix} 0 & AA_{12}\psi_p E_{p\mathrm{V}} & AA_{21}\psi_p E_{p\mathrm{H}} & 0 \\ AA_{11}\psi_p E_{p\mathrm{V}} & 0 & 0 & AA_{22}\psi_p E_{p\mathrm{H}} \\ AA_{11}\psi_p E_{p\mathrm{H}} & 0 & 0 & AA_{22}\psi_p E_{p\mathrm{V}} \\ 0 & AA_{12}\psi_p E_{p\mathrm{H}} & AA_{21}\psi_p E_{p\mathrm{V}} & 0 \end{bmatrix},$$

$$\mathbf{M}_{43} = \begin{bmatrix} 0 & -\gamma H\psi_f\psi_p BB_{12}E_{p\mathrm{V}} & -\gamma H\psi_f\psi_p BB_{21}E_{p\mathrm{H}} & 0 \\ -\gamma H\psi_f\psi_p BB_{11}E_{p\mathrm{V}} & 0 & 0 & -\gamma H\psi_f\psi_p BB_{22}E_{p\mathrm{H}} \\ -\gamma H\psi_f\psi_p BB_{11}E_{p\mathrm{H}} & 0 & 0 & -\gamma H\psi_f\psi_p BB_{22}E_{p\mathrm{V}} \\ 0 & -\gamma H\psi_f\psi_p BB_{12}E_{p\mathrm{H}} & -\gamma H\psi_f\psi_p BB_{21}E_{p\mathrm{V}} & 0 \end{bmatrix},$$

$$\mathbf{M}_{44} = \begin{bmatrix} Q_{11} & -Y_{12}\psi_p E_{p\mathrm{V}} & -Y_{21}\psi_p E_{p\mathrm{H}} & 0 \\ -Y_{11}\psi_p E_{p\mathrm{V}} & Q_{12} & 0 & -Y_{22}\psi_p E_{p\mathrm{H}} \\ -Y_{11}\psi_p E_{p\mathrm{H}} & 0 & Q_{21} & -Y_{22}\psi_p E_{p\mathrm{V}} \\ 0 & -Y_{12}\psi_p E_{p\mathrm{H}} & -Y_{21}\psi_p E_{p\mathrm{V}} & Q_{22} \end{bmatrix}. \tag{46}$$

The solution for the homogeneous case is given by $EE_{11} = 0$, or

$$Ra = \frac{R_{11}T_{11}}{S_{11}}. \tag{47}$$

For the weakly heterogeneous case we perturb this. We substitute

$$Ra = \frac{R_{11}T_{11}}{S_{11}}(1+S), \tag{48}$$

linearize for small S, and solve for S.

3. Results and discussion

The case of a regular porous medium corresponds to the limit as σ_f, ε and K_r tend to zero. The case of local thermal equilibrium corresponds to H tending to zero (or H tending to infinity with the Rayleigh number redefined, now expressed in terms of an effective thermal conductivity). Taking both limits produces the expression

$$S = -\frac{1}{63}\left[7(4\Delta_{f\mathrm{H}} - 5E_{f\mathrm{H}})^2 + 3(2\Delta_{f\mathrm{V}} - 5E_{f\mathrm{V}})^2\right]. \tag{49}$$

This leads to the critical value

$$Ra = 4\pi^2\left\{1 - \frac{64}{567\pi^2}\left[7(4\delta_{f\mathrm{H}} - 2.5\varepsilon_{f\mathrm{H}})^2 + 3(2\delta_{f\mathrm{V}} - 5\varepsilon_{f\mathrm{V}})^2\right]\right\}$$
$$\approx 39.48\{1 - 1.281(\delta_H - 0.625\varepsilon_H)^2 - 0.137(\delta_V - 2.5\varepsilon_V)^2\}. \tag{50}$$

This is the result obtained by Nield and Kuznetsov [4]. It shows that the effects of weak horizontal heterogeneity and vertical heterogeneity are each of second order in the property deviations and their combined contribution is of the order of the variances of the distributions for permeability and conductivity (which are here equal to $\delta_{f\mathrm{H}}^2 + \delta_{f\mathrm{V}}^2$ and $\varepsilon_{f\mathrm{H}}^2 + \varepsilon_{f\mathrm{V}}^2$, respectively). The effects of vertical heterogeneity and horizontal heterogeneity act independently at this order of approximation. (Product terms like $\delta_{f\mathrm{H}}\delta_{f\mathrm{V}}$ are absent in the last expression.) Since the expression in square brackets in Eq. (49) is positive definite, the heterogeneities lead to a reduction in the critical value of Ra for all combinations of horizontal and vertical heterogeneities and all combinations of permeability and conductivity heterogeneities. The effects of the horizontal permeability heterogeneity and the horizontal conductivity heterogeneity are at the first combination step subtractive.

In the general case one has

$$\begin{aligned} Ra = Ra_0(1 &+ C_{11\mathrm{H}}\delta_{f\mathrm{H}}^2 + C_{22\mathrm{H}}\delta_{p\mathrm{H}}^2 + C_{33\mathrm{H}}\varepsilon_{f\mathrm{H}}^2 \\ &+ C_{44\mathrm{H}}\varepsilon_{p\mathrm{H}}^2 + C_{12\mathrm{H}}\delta_{f\mathrm{H}}\delta_{p\mathrm{H}} + C_{34\mathrm{H}}\varepsilon_{f\mathrm{H}}\varepsilon_{p\mathrm{H}} \\ &+ C_{13\mathrm{H}}\delta_{f\mathrm{H}}\varepsilon_{f\mathrm{H}} + C_{24\mathrm{H}}\delta_{p\mathrm{H}}\varepsilon_{p\mathrm{H}} + C_{14\mathrm{H}}\delta_{f\mathrm{H}}\varepsilon_{p\mathrm{H}} \\ &+ C_{23\mathrm{H}}\delta_{p\mathrm{H}}\varepsilon_{f\mathrm{H}} + C_{11\mathrm{V}}\delta_{f\mathrm{V}}^2 + C_{22\mathrm{V}}\delta_{p\mathrm{V}}^2 + C_{33\mathrm{V}}\varepsilon_{f\mathrm{V}}^2 \\ &+ C_{44\mathrm{V}}\varepsilon_{p\mathrm{V}}^2 + C_{12V}\delta_{f\mathrm{V}}\delta_{p\mathrm{V}} + C_{34\mathrm{V}}\varepsilon_{f\mathrm{V}}\varepsilon_{p\mathrm{V}} \\ &+ C_{13\mathrm{V}}\delta_{f\mathrm{V}}\varepsilon_{f\mathrm{V}} + C_{24\mathrm{V}}\delta_{p\mathrm{V}}\varepsilon_{p\mathrm{V}} + C_{14\mathrm{V}}\delta_{f\mathrm{V}}\varepsilon_{p\mathrm{V}} \\ &+ C_{23\mathrm{V}}\delta_{p\mathrm{V}}\varepsilon_{f\mathrm{V}}). \end{aligned} \tag{51}$$

Table 1
Parameter values for the cases applicable to Table 2

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
ϕ	0.9999	0.4	0.4	0.4	0.4	0.6
ε	0.00001	0.4	0.4	0.4	0.4	0.6
A	1	1	1	1	1	1
Da_f	0	1	1	1	1	1
K_r	0.00001	0.0001	1	0.0001	0.0001	1
σ_f	0.00001	1	1	1	1	1
β	1	10	10	10	10	10
γ	1	1	1	1	10	1
H	0.00001	1	1	10	1	1

Table 2
Values of the Rayleigh number coefficients, defined by Eq. (51), for the various cases with parameter values given in Table 1

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Ra_0	39.48	1398.56	1255.46	1526.43	1180.83	1129.13
$C_{11\mathrm{H}}$	−1.281	0.032	0.031	0.032	0.032	0.031
$C_{22\mathrm{H}}$	0	0.000	0.003	0.000	0.000	0.003
$C_{33\mathrm{H}}$	−0.500	−0.194	−0.180	−0.139	−0.203	−0.184
$C_{44\mathrm{H}}$	0	−0.018	−0.024	−0.076	−0.047	−0.020
$C_{12\mathrm{H}}$	0	0.000	0.000	0.000	0.000	0.000
$C_{34\mathrm{H}}$	0	0.004	0.002	0.002	0.024	0.004
$C_{13\mathrm{H}}$	1.601	0.121	0.110	0.128	0.147	0.088
$C_{24\mathrm{H}}$	0	0.000	0.001	0.000	0.000	0.001
$C_{14\mathrm{H}}$	0	0.000	0.006	0.002	0.002	0.064
$C_{23\mathrm{H}}$	0	−0.023	0.010	−0.044	−0.050	0.006
$C_{11\mathrm{V}}$	−0.137	0.032	0.031	0.032	0.032	0.031
$C_{22\mathrm{V}}$	0	0.000	0.003	0.000	0.000	0.003
$C_{33\mathrm{V}}$	−0.858	−0.700	−0.660	−0.508	−0.752	−0.674
$C_{44\mathrm{V}}$	0	−0.054	−0.093	−0.300	−0.185	−0.078
$C_{12\mathrm{V}}$	0	0.000	0.000	0.000	0.000	0.000
$C_{34\mathrm{V}}$	0	0.022	0.019	0.038	0.120	0.023
$C_{13\mathrm{V}}$	0.686	0.120	0.108	0.126	0.145	0.085
$C_{24\mathrm{V}}$	0	0.000	0.001	0.000	0.000	0.001
$C_{14\mathrm{V}}$	0	0.000	0.006	0.002	0.001	0.006
$C_{23\mathrm{V}}$	0	−0.046	0.009	−0.044	−0.051	0.006

By plugging numerical parameter values into an algebraic expression obtained using Mathematica, we computed the Rayleigh number coefficients for a few representative cases, for the parameter values listed in Table 1. The computed values are given in Table 2. Case 1 approximates the case of a regular porous medium. For this case the computed results agree with the analytic formula given by Eq. (50). The other cases involve representative parameter values.

The results show that certain coefficients are generally small, and for practical purposes may be set equal to zero. These are $C_{22\mathrm{H}}$, $C_{12\mathrm{H}}$, $C_{24\mathrm{H}}$, $C_{22\mathrm{V}}$, $C_{12\mathrm{V}}$, $C_{24\mathrm{V}}$ and C_{14}. Also $C_{23\mathrm{H}}$, and $C_{23\mathrm{V}}$, are relatively small. The conclusion is that the effect of the hydrodynamic heterogeneity of the p-phase is generally small. This result could be expected. On the other hand, the thermal heterogeneity of the p-phase can be quite significant when the thermal diffusivity of the p-phase is relatively large.

4. Conclusions

We have examined the effects of both horizontal and vertical hydrodynamic (permeability) heterogeneity and thermal (conductivity) heterogeneity on the onset of convection in a horizontal layer of a saturated bidisperse porous medium uniformly heated from below using linear stability theory. For the case of weak heterogeneity we have carried out an analysis in terms of perturbation quantities representing the heterogeneity. We found that the effect of such heterogeneity on the critical value of the Rayleigh number Ra based on mean properties is of second order if the properties vary in a piecewise constant or linear fashion. The effects of horizontal heterogeneity and vertical heterogeneity are then comparable once the aspect ratio is taken into account, and to a first approximation are independent. A feature of the bidisperse porous medium is that the thermal heterogeneity of the p-phase can be quite significant when the thermal diffusivity of that phase is large relative to that of the f-phase.

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